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# EXISTENCE AND UNIQUENESS THEOREM FOR SOLUTIONS OF DYNAMIC PROBLEMS OF THE NONLINEAR THEORY OF ELASTICITY*

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The kernel of the elastic strain potential energy functional depends on the finite strain tensor invariants, while the functional itself is represented as a finite sum of homogeneous functionals of the displacement and is defined in the Sobolev space $(W_p^1(\Omega))^3$ $(p>2)$ /1,2/. A number of inequalities is set up which the strain potential energy functional and its Fréchet derivatives satisfy, and an existence and uniqueness theorem is proved for generalized solutions of dynamic problems of the nonlinear theory of elasticity in the phase space $(L_2(\Omega))^3 \times (W_p^1(\Omega))^3$ $(2<p<\infty)$.

Existence and uniqueness theorems were considered earlier for the solutions of dynamic problems of linear small-strain elasticity theory /3/ and of a class of nonlinear problems /4/.

**1. Properties of the potential elastic strain energy functional.** We give the potential strain energy functional in the form

$$\mathrm{E}[\mathbf{u}] = \int_\Omega e(\mathrm{I}_E, \mathrm{II}_E, \mathrm{III}_E)\,dx \quad (dx = dx_1\,dx_2\,dx_3), \quad e(\mathrm{I}_E, \mathrm{II}_E, \mathrm{III}_E) = \sum_{k=2}^{p} e_k(w), \quad w = (u_{11}, u_{12}, \ldots, u_{33}) \in R^9 \tag{1.1}$$

$$u_{ij} = \frac{\partial u_i}{\partial x_j}, \quad \forall\, i, j = 1, 2, 3, \quad \mathrm{E}_k[w] = \int_\Omega e_k(w)\,dx, \quad \mathbf{x} = (x_1, x_2, x_3) \in \Omega \subset R^3$$

Here $\Omega$ is the domain occupied by the body in the natural unstrained state, $\mathrm{I}_E, \mathrm{II}_E, \mathrm{III}_E$ are finite strain tensor invariants, $\mathbf{u}(\mathbf{x}, t)$ is the displacement vector, $e_k(w)$ are homogeneous functions of order $k$

$$e_k(w) = \sum_{|l|=k} a_l w^l, \quad e_k(\mu w) = \mu^k e_k(w), \quad \mu \in R^1, \quad k = 2, \ldots, p, \quad l = (l_1, \ldots, l_9), \quad w^l = w_1^{l_1} \ldots w_9^{l_9}, \quad |l| = \sum_{i=1}^{9} l_i. \tag{1.2}$$

where $l_i$ are nonnegative integers, and $a_l = a_{l_1, \ldots, l_9}$ are constants.

The functional (1.1) has the form mentioned in the case of a homogeneous isotropic medium. For an inhomogeneous, nonisotropic medium the coefficients $a_l$ in (1.2) depend on the point $\mathbf{x}$ and the orientation of the principal strain axes relative to the axes coupled to the medium. All the results obtained below for homogeneous and isotropic media will be valid in the general case if the coefficients $a_l(\mathbf{x})$ have upper and lower bounds in $\Omega$.

The domain of definition of the functional (1.1) is the Sobolev space $(W_p^1(\Omega))^3$ with the norm

$$\|\mathbf{u}\|_{p,1} = \Big(\sum_{i=1}^{3} \|u_i\|_{p,0}^p + \sum_{i,j=1}^{3} \|u_{ij}\|_{p,0}^p\Big)^{1/p}, \quad \|f\|_{p,0}^p = \int_\Omega |f|^p\,dx$$

**Theorem 1.** The functional (1.1) is bounded in $(W_p^1(\Omega))^3$.

The proof of Theorem 1 is based on using the Hölder inequality for several functions and estimates resulting from the theorem for embedding $L_p$ into $L_{|l|}$ for $|l|<p$ [5].

Let

$$G = \{\mathbf{u} : \mathbf{u} = \boldsymbol{\gamma} + (O - E)\,\mathbf{x}, \quad \mathbf{u}, \boldsymbol{\gamma} \in R^3, \; O \in SO(3)\}$$

be a rotational-displacement group in $R^3$, and let the conditions

$$\mathrm{E}[\mathbf{u}] = 0 \Leftrightarrow \mathbf{u} \in G; \quad \mathbf{u} \notin G \Rightarrow \mathrm{E}[\mathbf{u}] > 0 \tag{1.3}$$

be valid, which mean that the strain energy is zero for displacements of an elastic body as a solid, and positive in all the remaining cases. It follows that $p$ is even from the second condition in (1.3).

**Lemma 1.** The polynomial

$$P_p(y) = \sum_{|l|=p} a_l y^l, \quad y \in R^9 \tag{1.4}$$

cannot take on negative values.

If it is assumed that $P_p(y_0) = c < 0$ for $y = y_0$, then for $y = \mu y_0$ we obtain

$$E[\mu y_0] = \sum_{k=2}^{p} \mu^k E_k[y_0], \quad E_p[y_0] = c \operatorname{vol} \Omega < 0$$

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It is clear that for sufficiently large $\mu$ the functional $E[\mu y_0]$ can be made negative, which contradicts (1.3).

Lemma 2. Let the system of equations

$$P_p(y) = 0, \quad \operatorname{grad}_y P_p(y) = 0 \tag{1.5}$$

have no solutions. Then the functional $\mathrm{E}_p[\mathbf{u}]$ is positive definite

$$\mathrm{E}_p[\mathbf{u}] \geqslant c_1 \| w \|_{p,0}^p, \quad \forall\, w \in (L_p(\Omega))^9 \tag{1.6}$$

By virtue of the homogeneity of the polynomial $P_p(y)$, to prove the lemma it is sufficient to prove the inequality

$$P_p(y) \geqslant c_1 \sum_{i=1}^{9} y_i^p, \quad y \in S_1 = \left\{ y : \sum_{i=1}^{9} y_i^p = 1 \right\} \tag{1.7}$$

Since the sphere $S_1$ is compact in $R^9$, then the polynomial $P_p(y)$ takes a minimum value thereon. If the minimum of $P_p$ is positive in $S_1$, then (1.7) is proved. The minimum of the polynomial cannot be negative according to Lemma 1. There remains to examine the case when the minimum is zero. Since $P_p(y)$ is a differentiable function, then the vector $\operatorname{grad} P_p(y)$ at the minimum point should equal zero (its projection on the tangent hyperplane to $S_1$ vanishes, and the projection on the normal to $S_1$ equals zero since the polynomial remains zero along the normal because of homogeneity). The contradiction to conditions (1.5) proves the lemma.

Theorem 2. If the functional (1.1) ($p$ is even) satisfies conditions (1.3) and (1.5), then there exist constants $N > 0$, $c_2 > 0$ and the following inequality is valid

$$\mathrm{E}[\mathbf{u}] \geqslant c_2 \| w \|_{p,0}^p, \quad \| w \|_{p,0} > N \tag{1.8}$$

We have

$$\mathrm{E}[w] = \mu^p \left\{ \mathrm{E}_p[w^0] + \sum_{k=2}^{p-1} \mu^{k-p} \mathrm{E}_k[w^0] \right\}, \quad \mu = \| w \|_{p,0}, \quad w^0 = \mu^{-1} w$$

It follows from Theorem 1 that the homogeneous functionals

$$\mathrm{E}_k[w] = \int_\Omega e_k(w)\, dx$$

are bounded in $(W_p^1(\Omega))^3$. This means that for sufficiently large $\mu$ ($\mu > N$) the following estimate will be valid

$$\max_{\|w^0\|_{p,0}=1} \left\{ \sum_{k=2}^{p-1} \mu^{k-p} \mathrm{E}_k[w^0] \right\} < \frac{c_1}{2}$$

and furthermore, we obtain on the basis of (1.6)

$$\mathrm{E}[w] \geqslant 1/2\, c_1 \mu^p \| w^0 \|_{p,0}^p = 1/2\, c_1 \| w \|_{p,0}^p$$

which indeed proves (1.8).

Lemma 3. The gradient of the homogeneous functional $\mathrm{E}_k[w]$ satisfies the inequality

$$\| \nabla \mathrm{E}_k[w] \|_{k',0}^{k'} \leqslant M_k \| w \|_{k,0}^k, \quad \frac{1}{k} + \frac{1}{k'} = 1 \tag{1.9}$$

Remark. The functional $E_k[\mathbf{u}]$ can be examined either in the space $(W_k^1(\Omega))^3$ or the space $(L_k(\Omega))^3$ (in this case we shall write $E_k[w]$). Depending on this, the gradient $E_k$ will belong to the conjugate spaces $\nabla E_k[\mathbf{u}] \in (W_{k'}^{-1}(\Omega))^3$ and $\nabla E_k[w] \in (L_{k'}(\Omega))^3$, and the norms of the gradients are connected by the relationship

$$\| \nabla E_k[w] \|_{k',0} \geqslant \| \nabla E_k[\mathbf{u}] \|_{k',-1}$$

Proof of Lemma 3. According to (1.1) we have

$$\| \nabla E_k[w] \|_{k',0}^{k'} = \sum_{i=1}^{9} \left[ \int_\Omega \Big| \sum_{|m|=k} a_m m_i w^{m-1(i)} \Big|^{k'} dx \right]$$

Here the vector $m - 1(i)$ has the coordinates $(m_1, \ldots, m_i - 1, \ldots, m_9)$. Since

$$\Big| \sum_{s=1}^{n} z_s \Big|^{k'} \leqslant n^{k'-1} \sum_{s=1}^{n} | z_s |^{k'} \quad (k' > 1)$$

then

$$\| \nabla E_k[w] \|_{k',0}^{k'} \leqslant \sum_{i=1}^{9} \sum_{|m|=k} | a_m m_i |^{k'} 9^k \int_\Omega | w^{(m-1(i))k'} | dx \tag{1.10}$$

The integral in (1.10) is estimated by using the Hölder inequality for several functions /5/

$$\int_\Omega | w^{(m-1(i))k'} | dx \leqslant \prod_{s=1}^{9} \| w_s \|_{k,0}^{m_s - \delta_{is}} \quad (|m - 1(i)| k' = k)$$

We use the inequality

$$\prod_{s=1}^{n} |z_s| \leqslant \sum_{s=1}^{n} p_s^{-1} |z_s|^{p_s}, \quad \sum_{s=1}^{n} p_s^{-1} = 1$$

and obtain the estimate

$$\int_\Omega |w^{(m-1(i))k'}|\, dx \leqslant \sum_{s=1}^{9} \frac{m_s - \delta_{is}}{k} \| w_s \|_{k,0}^k \leqslant \| w \|_{k,0}^k$$

from which (1.9) follows with

$$M_k = \sum_{i=1}^{9} \sum_{|m|=k} 9^k | a_m m_i |^{k'}$$

**Theorem 3.** The gradient of the functional (1.1) satisfies the inequality

$$\| \nabla E [w] \|_{q,0}^q \leqslant N_1 \| w \|_{p,0}^q + N_2 \| w \|_{p,0}^p, \quad \frac{1}{p} + \frac{1}{q} = 1, \quad N_1 > 0, \quad N_2 > 0 \tag{1.11}$$

By using the embedding theorem $L_p \subset L_k$ $(k \leqslant p)$, and the resulting inequality $\| z \|_{k,0} \leqslant C_k \| z \|_{p,0}$, as well as Lemma 3, we arrive at the estimate

$$\| \nabla E [w] \|_{q,0} \leqslant \sum_{k=2}^{p} C_k M_k \| w \|_{k,0}^{k/k'}$$

Let us note that $k / k' = k - 1$ and the following inequality is true

$$\Big( \sum_{k=2}^{p} C_k M_k y^{k-1} \Big)^q \leqslant N_1 y^q + N_2 y^{(p-1)q}$$

for certain poisitive $N_1$, $N_2$, from which the assertion of the theorem follows.

**Corollary.** The inequality

$$\| \nabla E \ [w] \|_{q,0} \leqslant N_1' \| w \|_{p,0} + N_2' \| w \|_{p,0}^{p-1}$$

follows from the inequality (1.11), where $N_1'$, $N_2'$ are certain positive constants.

**Theorem 4.** The Lipschitz condition

$$\| \nabla E [w''] - \nabla E [w'] \|_{q,0} \leqslant L (h) \| w'' - w' \|_{p,0} \tag{1.12}$$

is valid for the gradient of the functional (1.1) if $\| w' \|_{p,0} < h$, $\| w'' \|_{p,0} < h$ $(h > 0)$, where $L(h)$ is a constant dependent only on $h$ and the domain of integration $\Omega$.

The proof of Theorem 4 is based on two lemmas.

**Lemma 4.** The second Fréchet derivative of the functional $E_k[w]$ satisfies the inequality

$$\| \nabla^2 E_k [w] \|_{(k)} = \sup \frac{(\nabla^2 E [w] z, v)}{\| z \|_{k,0} \| v \|_{k,0}} \leqslant B_k \| w \|_{k,0}^k, \quad B_k > 0, \quad z, v \in (L_k(\Omega))^9, \quad (\nabla^2 E [w] z, v) = \int_\Omega \sum_{i,j=1}^{9} \frac{\partial^2 e(w)}{\partial w_i \partial w_j} z_i v_j \, dx$$

**Lemma 5.** The second Fréchet derivative of the functional $E[w]$ satisfies the inequality

$$\| \nabla^2 E [w] \|_{(p)} \leqslant G_1 + G_2 \| w \|_{p,0}^{p-2}, \quad G_1 > 0, \quad G_2 > 0$$

The proofs of these lemmas are analogous to the proof of Lemma 3 and Theorem 3.

To prove Theorem 4 we consider the function

$$\Phi(\tau) = (\nabla E [w' + \tau (w'' - w')], v), \quad (\nabla E [w], v) = \int_\Omega \sum_{i=1}^{9} \frac{\partial e(w)}{\partial w_i} v_i \, dx$$

According to Lemma 5, its derivative satisfies the inequality

$$d\Phi / d\tau = (\nabla^2 E [w' + \tau (w'' - w')] (w'' - w'), v) \leqslant L (h) \| w'' - w' \|_{p,0} \| v \|_{p,0} \tag{1.13}$$

if $\| w'' \|_{p,0} < h$, $\| w' \|_{p,0} < h$ and $L(h) = G_1 + G_2 h^{p-2}$. Integrating (1.13) with respect to $\tau$ between zero and one, we arrive at the inequality

$$(\nabla E [w''] - \nabla E [w'], v) \leqslant L (h) \| w'' - w' \|_{p,0} \| v \|_{p,0}$$

Furthermore

$$\| \nabla E [w''] - \nabla E [w'] \|_{q,0} = \sup_{\|v\|_{p,0}=1} (\nabla E [w''] - \nabla E [w'], v) \leqslant L (h) \| w'' - w' \|_{p,0}$$

Q.E.D.

## 2. Existence theorem for the solutions.

The D'Alembert—Lagrange variational principle of the dynamical elasticity theory problem has the form /4/

$$(\mathbf{u}^{\cdot\cdot} + \nabla E [\mathbf{u}] - \mathbf{f}, \ \delta \mathbf{u}) - (\mathbf{F}, \ \delta \mathbf{u})_\Gamma = 0, \quad \forall \delta \mathbf{u} \in V \tag{2.1}$$

Here $\mathbf{f}$, $\mathbf{F}$ are the mass and surface forces, and $\Gamma = \partial\Omega$ is a differentiable manifold of dimensionality two satisfying the cone condition /5/. The elastic body is assumed homogeneous and isotropic with unit density.

The surface forces are given on a part of the boundary $\Gamma_F$, and the displacements $\mathbf{U}(\mathbf{x}, t)$ on a part of the boundary $\Gamma_U$, and $\Gamma = \Gamma_U \cup \Gamma_F$, $\Gamma_U \cap \Gamma_F = \varnothing$. The domain of definition of the functional $E[\mathbf{u}]$ is the Sobolev space $(W_p^1(\Omega))^3$. Then the traces of the function $\mathbf{u}(\mathbf{x}, t)$ on $\Gamma$ belong to the space of traces $(B_p^{1-1/p}(\Gamma))^3$, where $B_p^l(\Gamma)$ is the space of Besov that agrees with the Sobolev space for noninteger $l$ /5/. It hence follows that the displacement $\mathbf{U}(\mathbf{x}, t) \in (B_p^{1-1/p}(\Gamma))^3$ . and according to the theorem about traces, there exists a function $\mathbf{u}_0(\mathbf{x}, t)$ on $\Omega$ that belongs to $(W_p^1(\Omega))^3$ and satisfies the inequality

$$\|\mathbf{u}_0\|_{p,1} \leqslant d_1 \|\mathbf{U}^*\|_{p,1-1/p,\Gamma} \leqslant d_1 d_2 \|\mathbf{U}\|_{p,1-1/p,\Gamma}, \quad (d_1 > 0,\ d_2 > 0) \tag{2.2}$$

where $\mathbf{U}^*$ is the continuation of $\mathbf{U}$ on all of $\Gamma$, and $\|\cdot\|_{p,1-1/p,\Gamma}$ is the norm in $(B_p^{1-1/p}(\Gamma))^3$ /5/. The constants $d_1$, $d_2$ are independent of the functions in the inequality (2.2), and the boundary $\Gamma_U$ on $\Gamma$ (one-dimensional curve) satisfies the cone condition /5/.

The linear manifold $V \subset (W_p^1(\Omega))^3$, where

$$V = \{\mathbf{v} : \mathbf{v} \in (W_p^1(\Omega))^3,\ \mathbf{v}|_{\Gamma_U} = 0\}$$

is the configuration space of the mechanical system and the substitution $\mathbf{u} = \mathbf{u}_0 + \mathbf{v}$ reduces (2.1) to the form

$$(\mathbf{v}^{\cdot\cdot} + \nabla E[\mathbf{u}_0 + \mathbf{v}] - \mathbf{f}_0,\ \delta\mathbf{v}) - (\mathbf{F},\ \delta\mathbf{v})_\Gamma = 0,\ \ A\delta\mathbf{v} \in V,\ \ \mathbf{f}_0 = \mathbf{f} - \mathbf{u}^{\cdot\cdot}_0 \tag{2.3}$$

We speak below about the existence and uniqueness of solutions of the variational problem (2.3) in the phase space of the system $H \times V$ with the initial conditions

$$\mathbf{v}(\mathbf{x}, 0) = \mathbf{u}(\mathbf{x}, 0) - \mathbf{u}_0(\mathbf{x}, 0) \in V, \quad \mathbf{v}^\cdot(\mathbf{x}, 0) = \mathbf{u}^\cdot(\mathbf{x}, 0) - \mathbf{u}^\cdot_0(\mathbf{x}, 0) \in H,\ H = \{\mathbf{v}^\cdot : \mathbf{v}^\cdot \in (L_2(\Omega))^3,\ \mathbf{v}^\cdot|_{\Gamma_U} = 0\} \tag{2.4}$$

Let us consider a certain time segment $[0, T]$ and let us assume that

$$\mathbf{f}(\mathbf{x}, t) \in L_\infty(0, T; (L_2(\Omega))^3),\ \mathbf{F}(\mathbf{x}, t) \in L_\infty(0, T; (W_2^{1/2}(\Gamma))^3) \tag{2.5}$$

Conditions (2.5) constrain the intensity of the mass and surface forces in the time segment $[0, T]$

$$(\mathbf{f}, \mathbf{v}^\cdot) \leqslant \|\mathbf{f}\|_{2,0} \|\mathbf{v}^\cdot\|_{2,0} \leqslant K_1 \|\mathbf{v}^\cdot\|_{2,0},\ \ (\mathbf{F}, \mathbf{v}^\cdot)_\Gamma \leqslant \|\mathbf{F}\|_{2,1/2,\Gamma} \|\mathbf{v}^\cdot\|_{2,-1/2,\Gamma} \leqslant K_2 d_3 \|\mathbf{v}^\cdot\|_{2,0}$$
$$K_1 = \operatorname*{vrai\,max}_{0 \leqslant t \leqslant T} \|\mathbf{f}\|_{2,0},\quad K_2 = \operatorname*{vrai\,max}_{0 \leqslant t \leqslant T} \|\mathbf{F}\|_{2,1/2,\Gamma},\quad \|\mathbf{v}^\cdot\|_{2,-1/2,\Gamma} \leqslant d_3 \|\mathbf{v}^\cdot\|_{2,0} \tag{2.6}$$

The last inequality in (2.6) follows from the theorem on the traces of functions on a manifold /5/. Furthermore, let

$$\mathbf{U}, \mathbf{U}^\cdot \in (W_p^{1-1/p}(\Gamma))^3,\ \ \|\mathbf{U}\|_{p,1-1/p,\Gamma} \leqslant B_1,\ \ \|\mathbf{U}^\cdot\|_{p,1-1/p,\Gamma} \leqslant B_2 \tag{2.7}$$

$$\mathbf{U}^{\cdot\cdot} \in (W_2^{-1/2}(\Gamma)),\ \ \|\mathbf{U}^{\cdot\cdot}\|_{2,-1/2,\Gamma} \leqslant B_3,\ \ \forall t \in [0, T] \tag{2.8}$$

In combination with inequality (2.2) the conditions (2.7) assure the continuation of $\mathbf{U}$, $\mathbf{U}^{\cdot\cdot}$ on the whole manifold $\Gamma$ and the estimates

$$\|\mathbf{u}_0\|_{p,1} \leqslant d_1 d_2 B_1,\ \ \|\mathbf{u}_0^\cdot\|_{p,1} \leqslant d_1 d_2 B_2 \tag{2.9}$$

It follows from condition (2.8) that $\mathbf{u}_0^{\cdot\cdot} \in (L_2(\Omega))^3$ and

$$\|\mathbf{u}_0^{\cdot\cdot}\|_{2,0} \leqslant d_4 B_3,\quad d_4 > 0 \tag{2.10}$$

**Theorem.** Let a homogeneous isotropic elastic medium occupy a domain $\Omega$ with smooth boundary $\Gamma$ in the natural state, let the potential of the elastic forces be given by (1.1), let the external forces and displacements on parts of the boundary satisfy the conditions (2.5), (2.7), (2.8), then (2.3) has the following solution

$$\mathbf{v}(\mathbf{x}, t) \in L_\infty(0; T; V),\ \ \mathbf{v}^\cdot(\mathbf{x}, t) \in L_\infty(0, T; H) \tag{2.11}$$

for the initial conditions (2.4). The proof of the theorem consists of the following fundamental steps: construction of approximate solutions by the Galerkin method, proof of their boundedness and proof of the fact that the limit function satisfies (2.3) and the initial conditions (2.4) /4/.

**Construction of the approximate solutions.** Let $\{\boldsymbol{\varphi}_k\}_{k=1}^\infty$ be an orthogonal basis in $H$ satisfying the conditions $\boldsymbol{\varphi}_1(\mathbf{x}) = \mathbf{v}(\mathbf{x}, 0) / \|\mathbf{v}(\mathbf{x}, 0)\|_{p,1}$, $\|\boldsymbol{\varphi}_k\|_{p,1} = 1$. This is possible since the space $W_p^1(\Omega)$ $(p > 2)$ is embedded in $L_2(\Omega)$ and compact therein. Let us define the approximate solution $\mathbf{v}^{(n)}(\mathbf{x}, t)$ as a solution of the equation

$$(\mathbf{v}^{\cdot\cdot(n)} + \nabla E[\mathbf{u}_0 + \mathbf{v}^{(n)}] - \mathbf{f}_0, \delta\mathbf{v}) - (\mathbf{F}, \delta\mathbf{v})_\Gamma = 0,\ \ \forall\delta\mathbf{v} \in V^{(n)} \tag{2.12}$$

that satisfies the initial conditions $\mathbf{v}^{(n)}(\mathbf{x}, 0) = \mathbf{v}(\mathbf{x}, 0)$, $\mathbf{v}^{\cdot(n)}(\mathbf{x}, 0) = P_n \mathbf{v}^\cdot(\mathbf{x}, 0)$, where $P_n$ is the projection operator of $H^{(n)}$ in $H$. The spaces $V^{(n)}$ and $H^{(n)}$ are linear spans of the vectors $\{\boldsymbol{\varphi}_k\}_{k=1}^n$ with the norms $W_p^1$ and $L_2$, respectively.

Using the representation $$\mathbf{v}^{(n)}(\mathbf{x}, t) = \sum_{s=1}^{n} q_{sn}(t)\,\boldsymbol{\varphi}_s(\mathbf{x})$$

and setting $\delta\mathbf{v} = \boldsymbol{\varphi}_r(\mathbf{x})$ $(r = 1, \ldots, n)$ into (2.12), we obtain a system of $2n$ ordinary differential equations equivalent to (2.12)

$$\dot{q}_{rn} = p_{rn}\|\boldsymbol{\varphi}_r\|_{2,0}^{-1}, \quad p_{rn} = \Phi_{rn}(q_{1n}, \ldots, q_{nt}, t), \quad r = 1, \ldots, n \tag{2.13}$$

$$\Phi_{rn} = \|\boldsymbol{\varphi}_r\|_{2,0}^{-1}[(\mathbf{f}_0 - \nabla \mathrm{E}[\mathbf{u}_0 + \mathbf{v}^{(n)}], \boldsymbol{\varphi}_r) + (\mathbf{F}, \boldsymbol{\varphi}_r)_\Gamma]$$

We estimate the right sides of (2.13) in the norms $L_p$ and $L_2$, respectively, we use the inequality (1.12), and we arrive at the deduction that the Lipschitz condition with constant $Z(n, h) = \max(1, nL(h))(\min_{1\leqslant r\leqslant n}\|\boldsymbol{\varphi}_r\|_{2,0})^{-1}$ is satisfied for them if $\|D(\mathbf{v}^{(n)\prime} + \mathbf{u}_0)\|_{p,0} < h$ and $\|D(\mathbf{v}^{(n)\prime\prime} + \mathbf{u}_0)\|_{p,0} < h$.
The operator $D$ denotes partial derivatives taken with respect to all the variables and the vector $D\mathbf{v}^{(n)} \in (L_p(\Omega))^9$. According to (2.9), these conditions will be satisfied if

$$\mathbf{v}^{(n)\prime},\ \mathbf{v}^{(n)\prime\prime} \in S_b = \{\mathbf{v}^{(n)}:\ \|\mathbf{v}^{(n)}\|_{p,1} < b = h - d_1 d_2 B_1\}$$

Taking account of the corollary from Theorem 3 and the inequalities (2.6), (2.9), (2.10), the right sides of (2.13) satisfy the inequality

$$\Big(\sum_{r=1}^{n} p_{rn}^p \|\boldsymbol{\varphi}_r\|_{2,0}^{-p}\Big)^{1/p} + \Big(\sum_{r=1}^{n}\Phi_{rn}^2\Big)^{1/2} \leqslant M(n, h)$$

$$M(n, h) = (\min_{1\leqslant r\leqslant n}\|\boldsymbol{\varphi}_r\|_{2,0})^{-1}[h - d_1 d_2 B_1 + n(N_1' h + N_2' h^{p-1}) + n\max_{1\leqslant r\leqslant n}\|\boldsymbol{\varphi}_r\|_{2,0}(K_1 + K_2 + d_4 B_3)]$$

under the condition that

$$\|\mathbf{p}_n\|_{2,0} + \|\mathbf{v}^{(n)}\|_{p,1} < h - d_1 d_2 B_1$$

By the existence and uniqueness theorem for solutions, the system (2.13) has a unique solution in the time interval /6/

$$T_1^{(n)} = \min(Z^{-1}(n, h),\ (h - d_1 d_2 B_1 - a_1)M^{-1}(n, h)), \quad a_1 = \|\mathbf{v}^{\cdot}(\mathbf{x}, 0)\|_{2,0} + \|\mathbf{v}(\mathbf{x}, 0)\|_{p,1}$$

Let us note that $T_1^{(n)} \to 0$ as $n \to \infty$.

**Boundedness of the solutions.** We replace $\delta\mathbf{v}$ in (2.12) by $\mathbf{v}^{\cdot(n)}$ and we integrate the equality obtained between 0 and $t$

$$\frac{1}{2}\|\mathbf{v}^{\cdot(n)}\|_{2,0}^2 + \mathrm{E}[\mathbf{u}_0 + \mathbf{v}^{(n)}] = L_1^{(n)} + \int_0^t [(\nabla \mathrm{E}[\mathbf{u}_0 + \mathbf{v}^{(n)}], \mathbf{u}_0^{\cdot}) + (\mathbf{f}_0, \mathbf{v}^{\cdot(n)}) + \tag{2.14}$$

$$(\mathbf{F}, \mathbf{v}^{\cdot(n)})_\Gamma]\,d\tau, \quad L_1^{(n)} = \frac{1}{2}\|\mathbf{v}^{\cdot(n)}(\mathbf{x}, 0)\|_{2,0}^2 + \mathrm{E}[\mathbf{u}_0(\mathbf{x}_0, 0) + \mathbf{v}(\mathbf{x}, 0)] \leqslant \frac{1}{2}\|\mathbf{v}^{\cdot}(\mathbf{x}, 0)\|_{2,0}^2 + \mathrm{E}[\mathbf{u}_0(\mathbf{x}, 0) + \mathbf{v}(\mathbf{x}, 0)] = L_1$$

Estimating the right side of (2.14) by using the inequalities (2.6)—(2.10) and (1.11), we arrive at the inequality

$$\frac{1}{2}\|\mathbf{v}^{\cdot(n)}\|_{2,0}^2 + \mathrm{E}[\mathbf{u}_0 + \mathbf{v}^{(n)}] \leqslant L_2(t) + \int_0^t [A_1\|D(\mathbf{u}_0 + \mathbf{v}^{(n)})\|_{p,0}^q + A_2\|D(\mathbf{u}_0 + \mathbf{v}^{(n)})\|_{p,0}^p + A_3\|\mathbf{v}^{\cdot(n)}\|_{2,0}^2]\,d\tau \tag{2.15}$$

$$A_1 = N_1/q, \quad A_2 = N_2/q, \quad A_3 = 1 + {}^1/_2 d_4^2, \quad L_2(t) = L_1 + t[p^{-1}(d_1 d_2 B_2)^p + {}^1/_2(K_1^2 + K_2^2 + d_4^2 B_3^2)]$$

We examine two cases. Let $\|D(\mathbf{u}_0 + \mathbf{v}^{(n)})\|_{p,0} < N$, then $0 \leqslant \mathrm{E}[\mathbf{u}_0 + \mathbf{v}^{(n)}] < L_3$ according to Theorem 1, and (2.15) is converted into the inequality

$$\|\mathbf{v}^{\cdot(n)}\|_{2,0}^2 \leqslant L_4 + 2A_3\int_0^t \|\mathbf{v}^{\cdot(n)}\|_{2,0}^2\,d\tau, \quad L_4 = 2[L_2(T) + A_1 N^q T + A_2 N^p T]$$

On the basis of the Grenouille inequality /3/

$$\|\mathbf{v}^{\cdot(n)}\|_{2,0}^2 \leqslant L_4 \exp(2A_3 t) \tag{2.16}$$

When $\|D(\mathbf{u}_0 + \mathbf{v}^{\cdot(n)})\|_{p,0} > N$, according to Theorem 2 (the inequality (1.8))

$$\frac{1}{2}\|\mathbf{v}^{\cdot(n)}\|_{2,0}^2 + c_2 z \leqslant L_2(T) + \int_0^t (A_1 z^{q/p} + A_2 z + A_3\|\mathbf{v}^{\cdot(n)}\|_{2,0}^2)\,d\tau, \quad z = \|D(\mathbf{u}_0 + \mathbf{v}^{(n)})\|_{p,0}^p \tag{2.17}$$

Since $q/p < 1$, and $z > N^p$, then $z^{q/p} < A_4 z$. If we use the notation $\min({}^1/_2, c_2) = c_3$, $\max(A_1 A_4 + A_2, A_3) = A_5$, $\|\mathbf{v}^{\cdot(n)}\|_{2,0}^2 + z = y$, then we obtain from (2.17)

$$c_2 y(t) \leqslant L_2(T) + A_5\int_0^t y(\tau)\,d\tau$$

and according to the Grenouille inequality

$$y(t) \leqslant c_2^{-1} L_2(T)\exp(c_2^{-1} A_5 t) \tag{2.18}$$

The boundedness of $\|\mathbf{v}^{(n)}\|_{2,0}$ follows from the boundedness of $\|\mathbf{v}^{\cdot(n)}\|_{2,0}$ since /3/

$$\| \mathbf{v}^{(n)} \|_{2,0}^2 \leqslant 2 \| \mathbf{v}(\mathbf{x}, 0) \|_{2,0}^2 + c_4 \int_0^t \| \mathbf{v}^{\cdot(n)} \|_{2,0}^2 d\tau \quad (c_4 > 0) \tag{2.19}$$

On the other hand, on the basis of the miltiplicative inequality /5/

$$\| \mathbf{v}^{(n)} \|_{p,0} \leqslant c_5 (\| \mathbf{v}^{(n)} \|_{2,0} + \| D\mathbf{v}^{(n)} \|_{p,0}) \quad (c_5 > 0) \tag{2.20}$$

Combining the results (2.16), (2.18)—(2.20), we arrive at the deduction that there exist $Q > 0$ and $\alpha > 0$ and the following estimate is valid

$$\| \mathbf{v}^{\cdot(n)} \|_{2,0} + \| \mathbf{v}^{(n)} \|_{p,1} \leqslant Qe_\alpha^{\ t} \tag{2.21}$$

The constants $Q$ and $\alpha$ in the inequality (2.21) are independent of the number $n$.

As has been shown above, the solution of the system (2.13) exists in the time segment $[0, T_1^{(n)}]$. Let us examine the question of continuation of the solution in the time segment $[0, T]$. If

$$a_2 = \| \mathbf{v}^{\cdot(n)} (\mathbf{x}, \ T_1^{(n)}) \|_{2,0} + \| \mathbf{v}^{(n)} (\mathbf{x}, \ T_1^{(n)}) \|_{p,1} < h - d_1 d_2 B_1$$

then the solution can be continued in the segment $[T_1^{(n)}, T_2^{(n)}]$, where

$$T_2^{(n)} - T_1^{(n)} = \min (Z^{-1}(n, h), (h - d_1 d_2 B_1 - a_2) M^{-1}(n, h))$$

The process of continuing the solution can be repeated until $h - d_1 d_2 B_1 - a_k$ becomes negative. Taking account of the growth estimate (2.21), we arrive at the deduction that the solution will exist in the time segment $[0, T']$, where $T'$ satisfies the equality

$$h - d_1 d_2 B_1 - Q \exp (\alpha T') = 0$$

Selecting $h$ sufficiently large (this selects the domain in the system phase space), the existence of the solution can be assured in the segment $[0, T]$ for any number $n$. All the solutions $(\mathbf{v}^{\cdot(n)}, \mathbf{v}^{(n)})$ of the system (2.13) are bounded in the space $L_\infty (0, T; H \times V)$.

Convergence of the successive approximations. We use the property of bounded sequences in functional spaces: Out of all the bounded sequences in a reflective Banach space, a weakly convergent subsequence can be selected /7/:

$$(\mathbf{v}^{\cdot(n)}, \mathbf{v}^{(n)}) \rightarrow (\mathbf{v}^{\cdot}, \mathbf{v}) \quad \text{weakly in} \quad L_\infty (0, T; H \times V)$$

Here $n$ runs through a certain subsequence of natural numbers.

It is shown by a method analogous to that mentioned in /4/ that the limit function satisfies equation (2.3) and the initial conditions (2.4), and the equation

$$\mathbf{v}^{\cdot\cdot} + \nabla \mathrm{E} [\mathbf{u}_0 + \mathbf{v}] = \boldsymbol{\Phi}, \ (\boldsymbol{\Phi}, \boldsymbol{\psi}) = (\mathbf{f}_0, \boldsymbol{\psi}) + (\mathbf{F}, \boldsymbol{\psi})_\Gamma, \ \forall \boldsymbol{\psi} \in V$$

is understood in the sense of distributions in the segment $[0, T]$ with values in $V'$, a space conjugate to the configuration space $V$.

3. Uniqueness of the solutions. We formulate two theorems establishing the uniqueness of the solutions.

Theorem (stationary case). Let the solution $\mathbf{v}$ of equation (2.3) be such that $\mathbf{u} = \mathbf{u}_0 + \mathbf{v}$ is independent of the time, and the functional $\mathrm{E}[\mathbf{u}]$ is convex.

$$\mathrm{E} [\mathbf{u} + \Delta\mathbf{v}] - \mathrm{E} [\mathbf{u}] - (\nabla \mathrm{E} [\mathbf{u}], \ \Delta\mathbf{v}) \geqslant \beta \| \Delta\mathbf{v} \|_{p,1}^2, \ (\beta > 0, \ \| \Delta\mathbf{v} \|_{p,1} < h_1, \ h_1 > 0) \tag{3.1}$$

Then this solution is unique.

Remark. Condition (3.1) can be replaced by a condition on the second Fréchet variation of the functional $\mathrm{E}[\mathbf{w}]$

$$\| \mathbf{w} - \mathbf{u} \|_{p,1} < h_1 \ (h_1 > 0), \ (\nabla^2 \mathrm{E} [\mathbf{w}] \Delta\mathbf{v}, \ \Delta\mathbf{v}) \geqslant 2\beta \| \Delta\mathbf{v} \|_{p,1}^2 \tag{3.2}$$

Conditions (3.1) and (3.2) are conditions for local convexity of the functional $\mathrm{E}[\mathbf{u}]$. The proof of the theorem is analogous to that indicated in /4/.

Theorem (dynamic case). If the functional $\mathrm{E}[\mathbf{u}]$ satisfies the condition (3.1) and $\mathbf{v}^{\cdot} \in L_\infty (0, T; V)$, then the solution $\mathbf{v}(\mathbf{x}, t)$ of the variational equation (2.3) is unique.

The following lemma is used to prove this theorem: The third Fréchet differential of the functional $\mathrm{E}[\mathbf{u}]$ satisfies the condition

$$(\nabla^3 \mathrm{E} [w] (z_1, \ z_2), \ z_3) \leqslant (D_1 + D_2 \| w \|_{p,0}^{p-3}) \ \| z_1 \|_{p,0} \| z_2 \|_{p,0} \times \| z_3 \|_{p,0}, \ w, \ z_i \in (L_p(\Omega))^9 \quad (i = 1, 2, 3) \tag{3.3}$$

where $D_1$, $D_2$ are positive constants.

Proof of the lemma is analogous to the proof of Lemma 3 and Theorem 3. Then for $w = D(\mathbf{u}_0 + \mathbf{v})$

$$(\nabla^3 \mathrm{E} [w] (\Delta\mathbf{v}, \ \Delta\mathbf{v}), \ \mathbf{u}_0^{\cdot} + \mathbf{v}^{\cdot}) \leqslant M \| \Delta\mathbf{v} \|_{p,1}^2 \ (M > 0)$$

and subsequent proof of the theorem is carried out by the scheme to prove the uniquenss theorem in the dynamic case in /4/.

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